A brief survey of uncertainty quantification

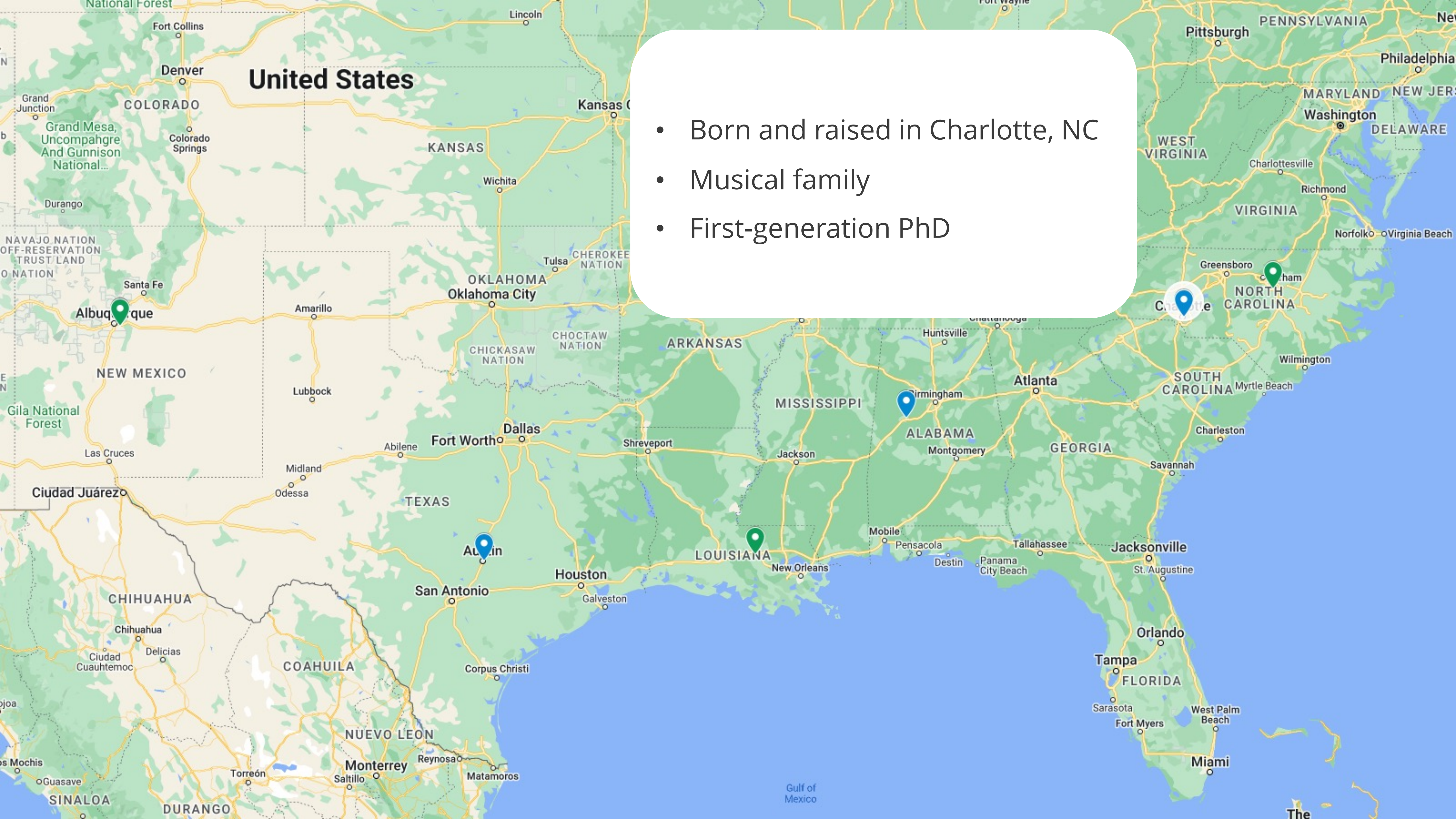
Teresa Portone

November 1, 2022

University of Alabama Mathematics Colloquium

Sandia National Laboratories is a multimission laboratory managed and operated by National Technology & Engineering Solutions of Sandia, LLC, a wholly owned subsidiary of Honeywell International Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA0003525.

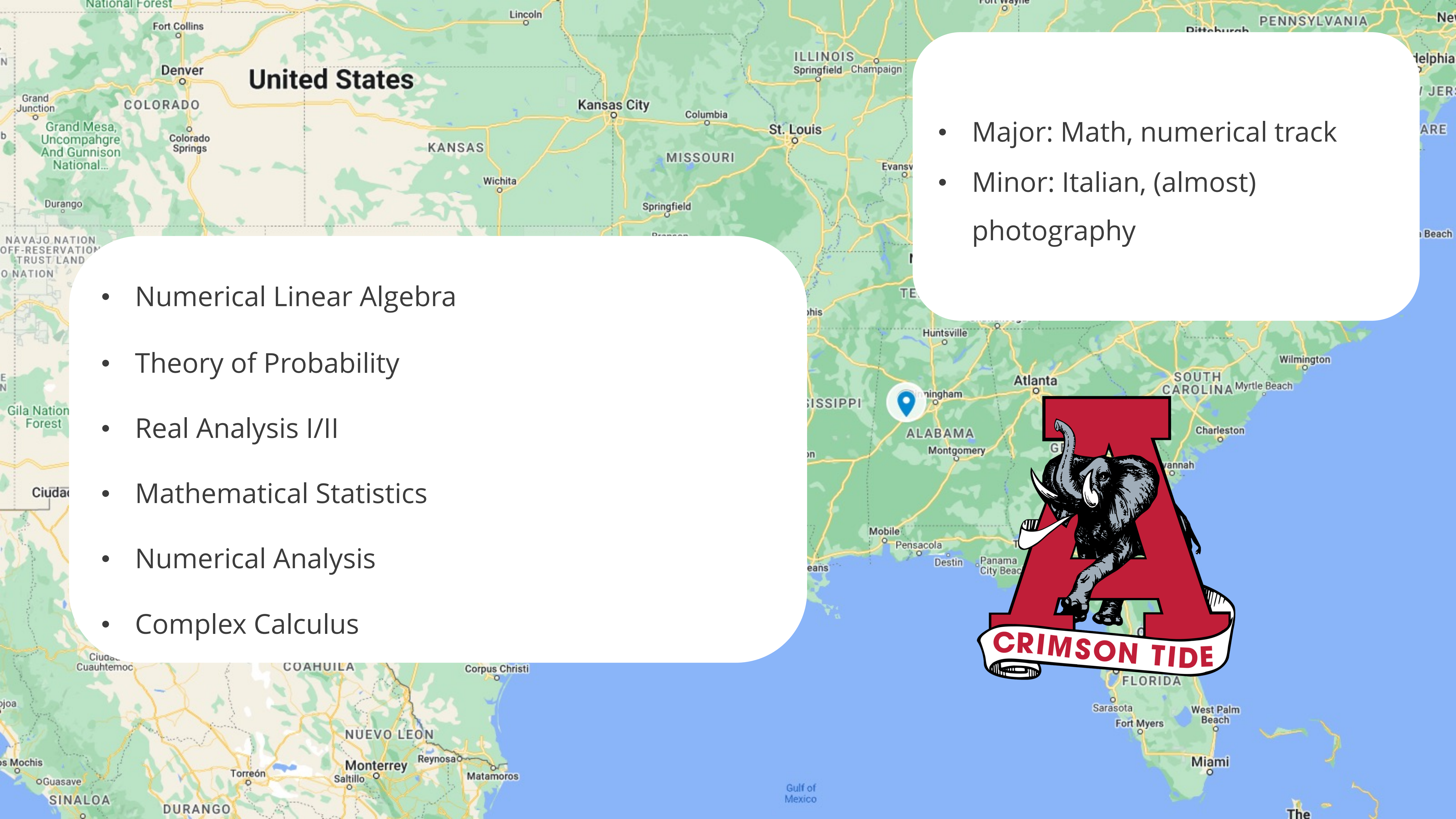
SAND2022-15190 PE

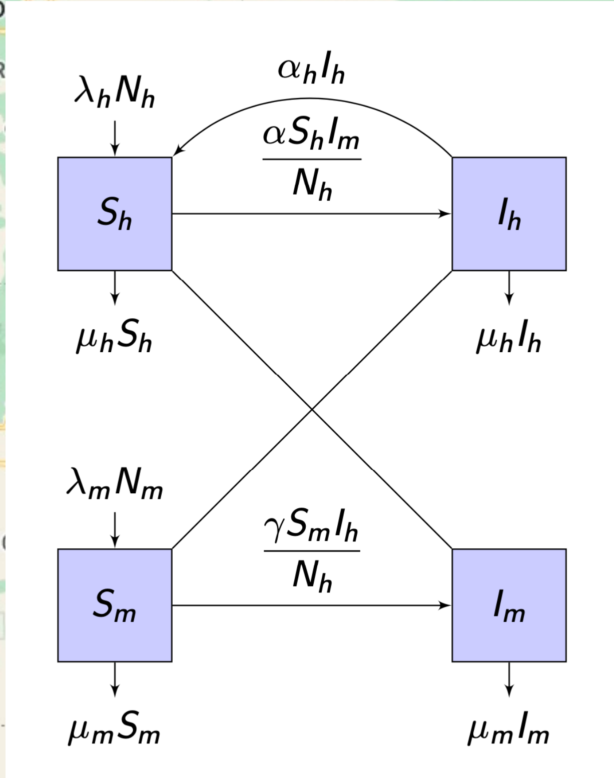


- Born and raised in Charlotte, NC
- Musical family
- First-generation PhD

- Major: Math, numerical track
- Minor: Italian, (almost) photography

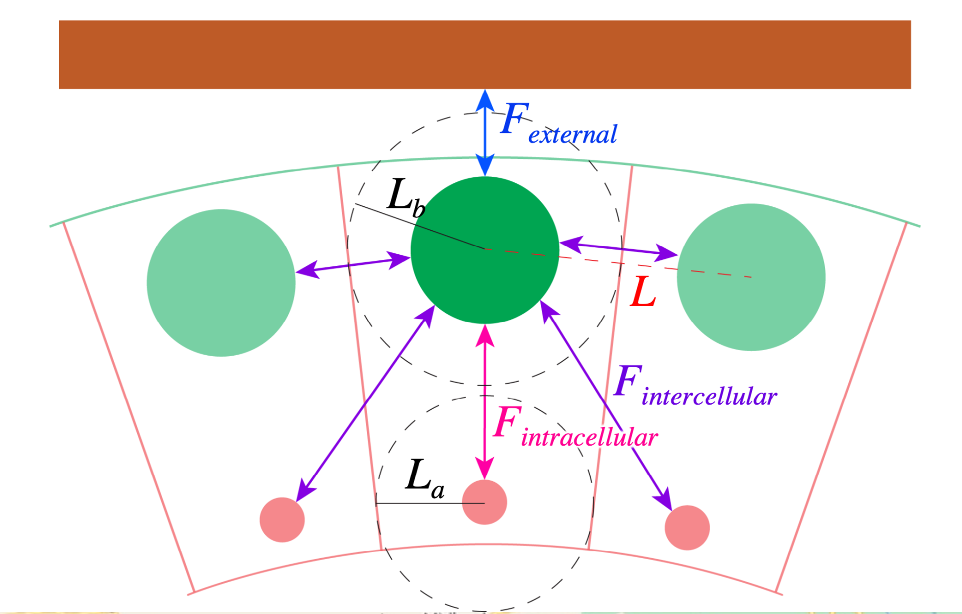
- Numerical Linear Algebra
- Theory of Probability
- Real Analysis I/II
- Mathematical Statistics
- Numerical Analysis
- Complex Calculus

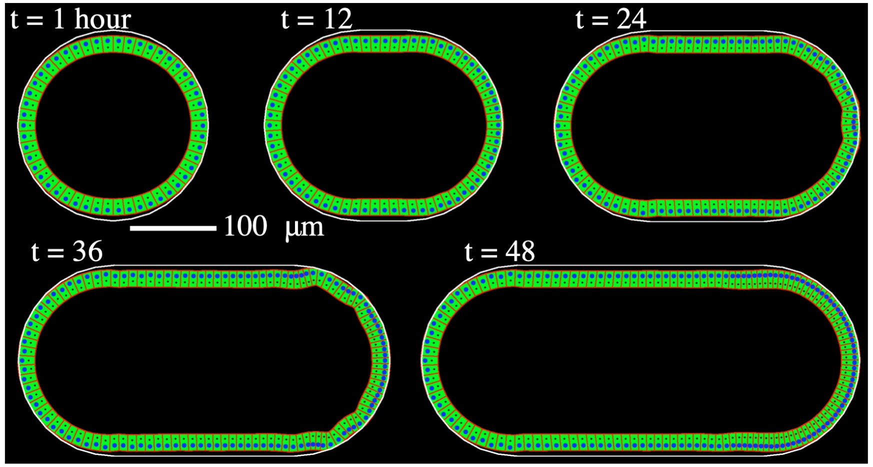




- VIGRE, Summer 2011
- Mathematical epidemiology project (Malaria)

- REU, Summer 2012
- Numerical model for embryonic tube formation





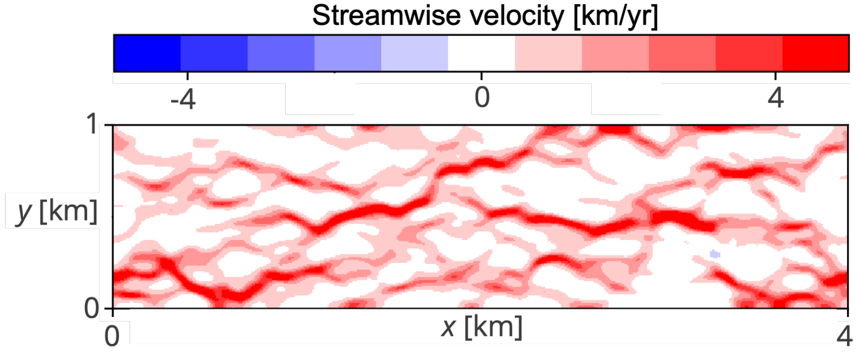
NC STATE UNIVERSITY

- PhD Computational Science, Engineering, and Mathematics
- Thesis “Representing Model-Form Uncertainty from Missing Microstructural Information”

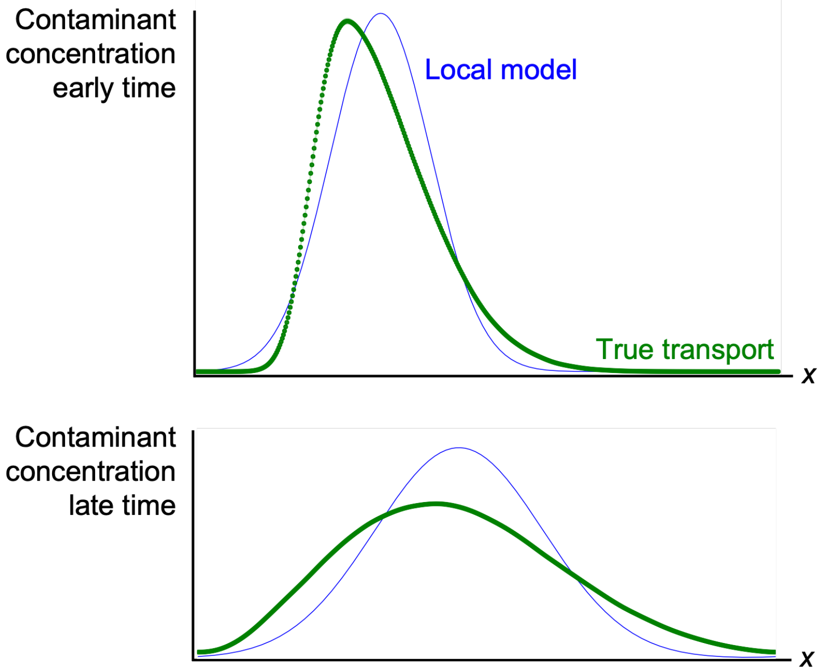
The University of Texas at Austin

Oden Institute for Computational Engineering and Sciences

High-velocity “streaks” cause nonlocality in contaminant transport.



Traditional upscaled models ignore nonlocality.

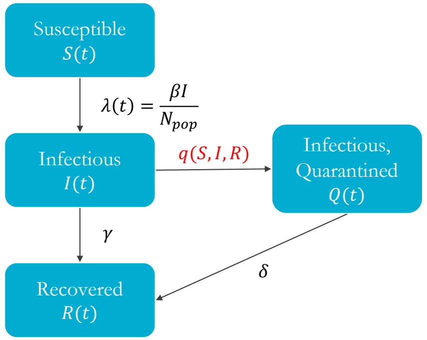


- Optimization and Uncertainty Quantification department
  - Interned Fall 2017
  - Joined as staff January 2020

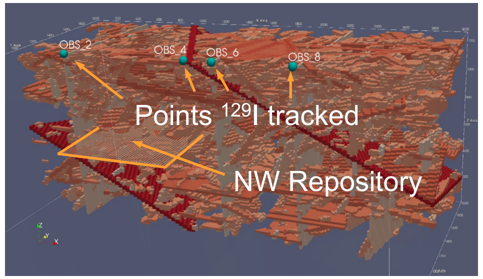
Sandia National Laboratories

Metal yield modeling under impact

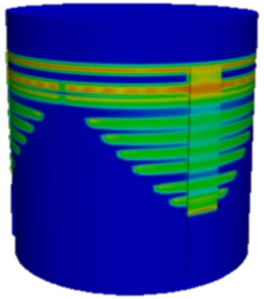
Disease modeling



Nuclear waste repository modeling



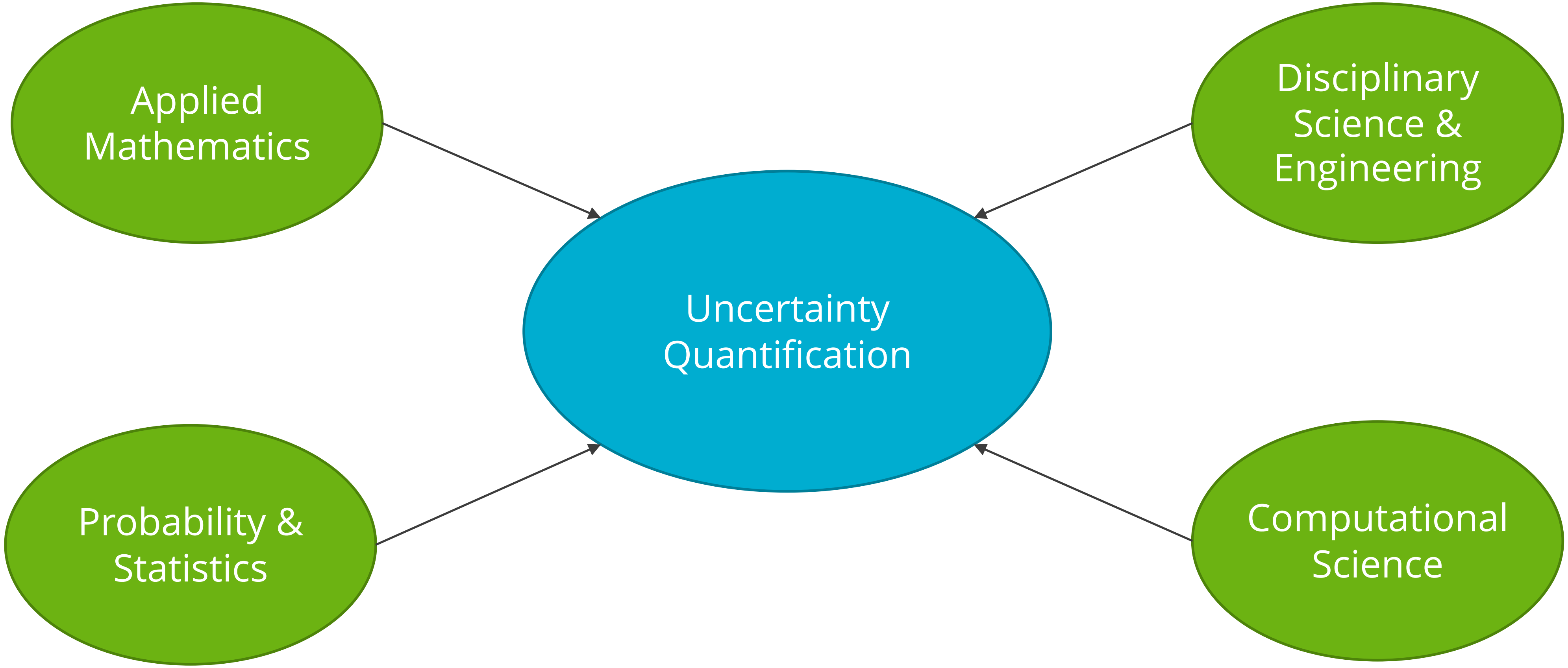
Thermally activated battery modeling



What is uncertainty quantification (UQ)?

My definition:

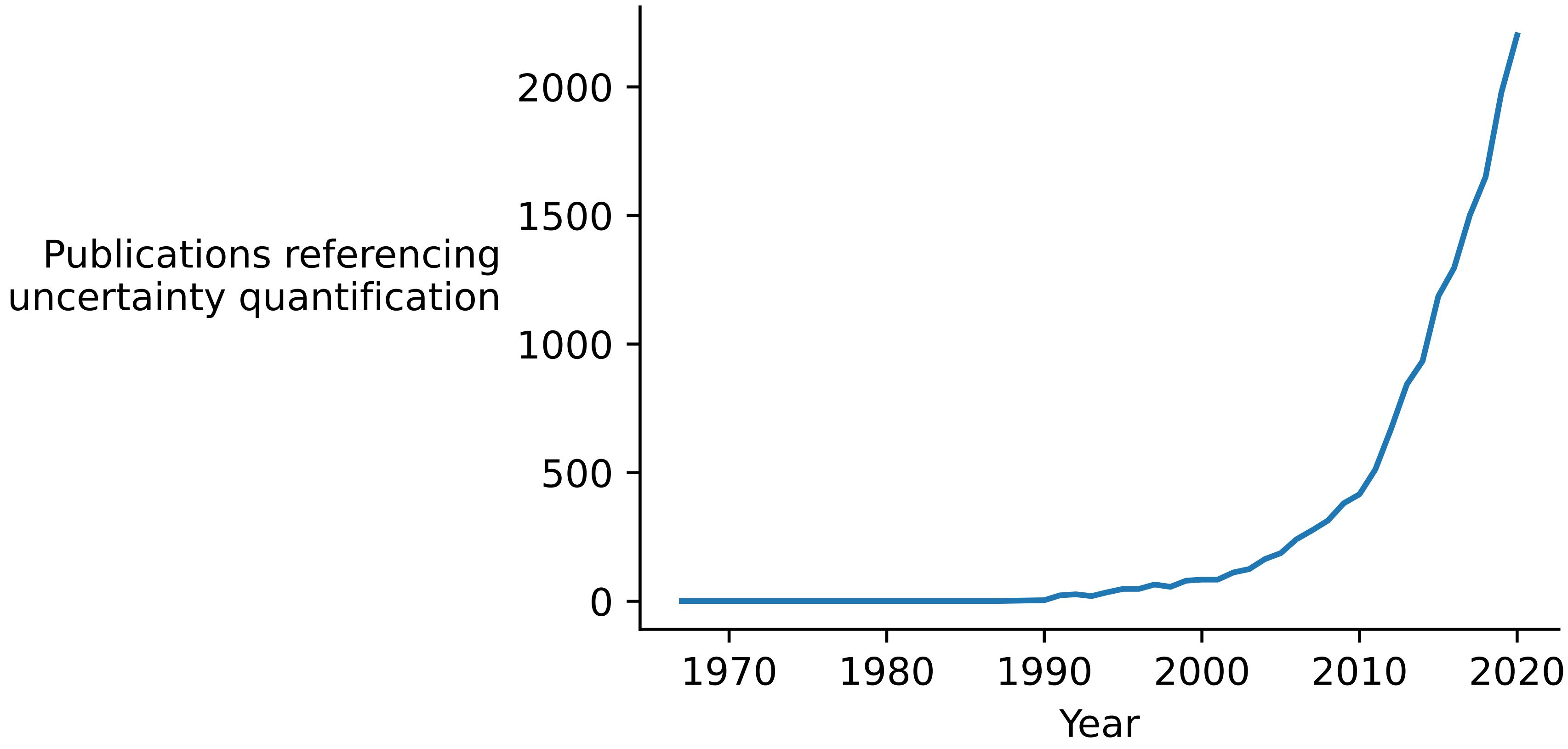
The science of characterizing, quantifying, and reducing uncertainties in mathematical models.



General references: [1-3]

UQ has taken off in the last couple decades

“Uncertainty quantification is both a new field and one that is as old as the disciplines of probability and statistics.” (Smith 2013) [1]



Source: Web of Science

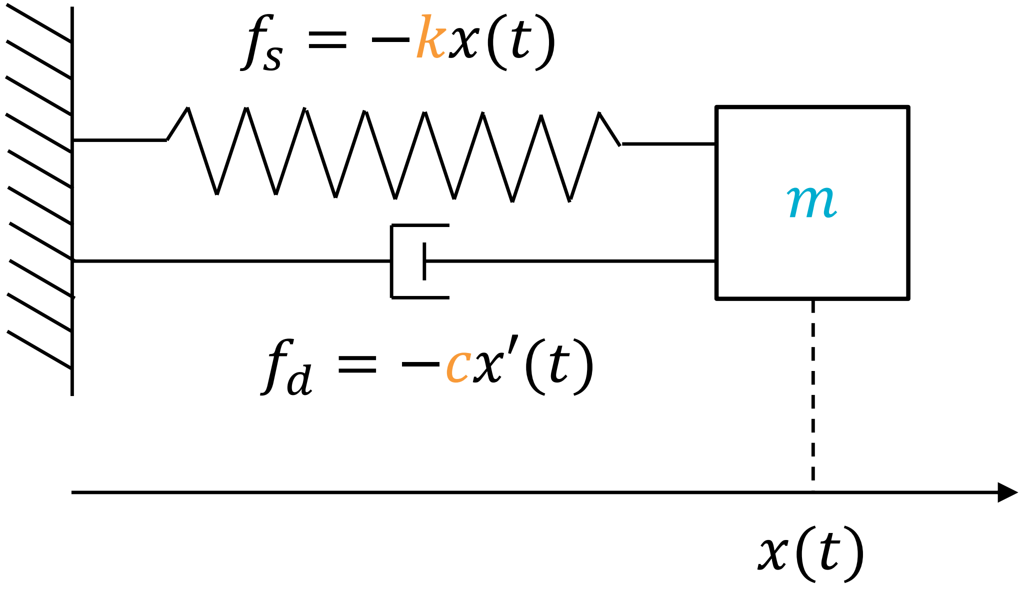
What is uncertainty?

Inability to assign an exact value to a modeled quantity.

How does uncertainty arise?

Intrinsic variability in a modeled quantity

Lack of precise knowledge of a modeled quantity

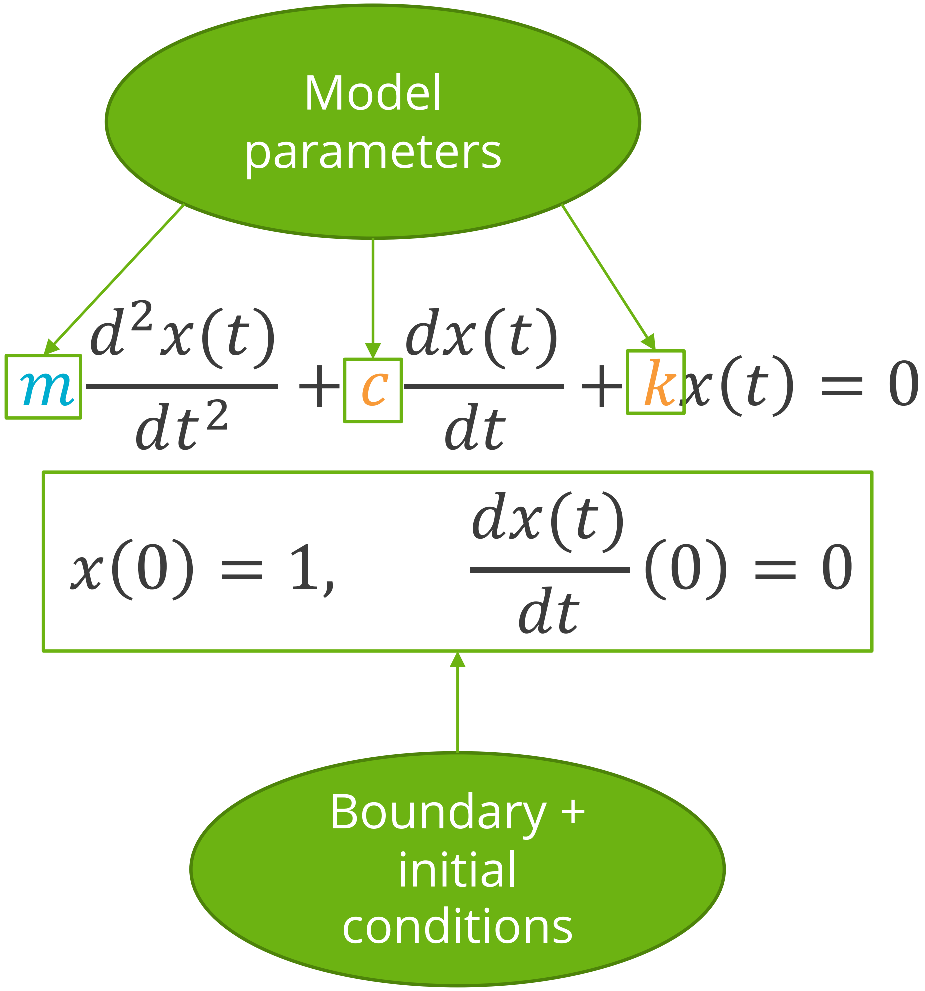


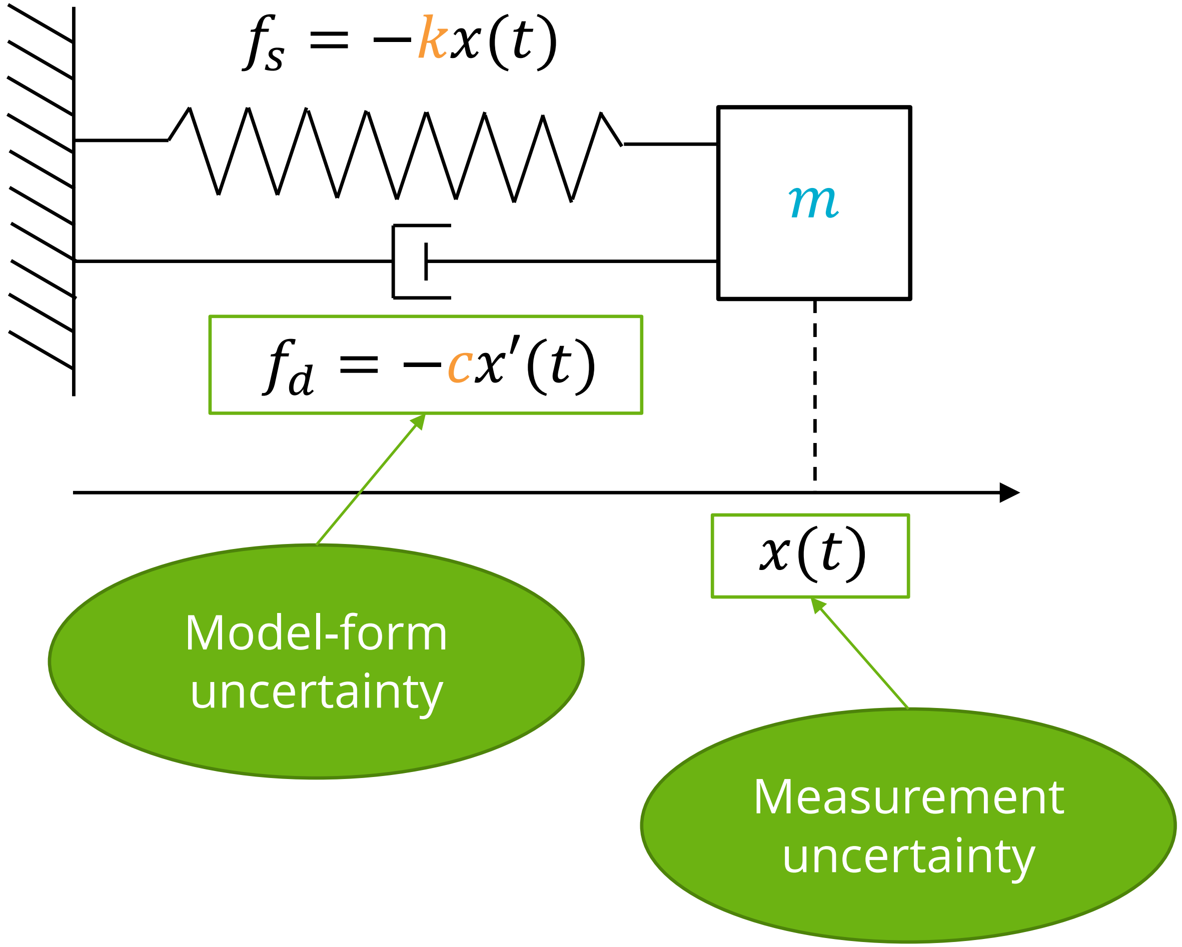
$$f_s + f_d \equiv F = ma \equiv m\frac{d^2x(t)}{dt^2}$$

$$m\frac{d^2x(t)}{dt^2} + c\frac{dx(t)}{dt} + kx(t) = 0$$

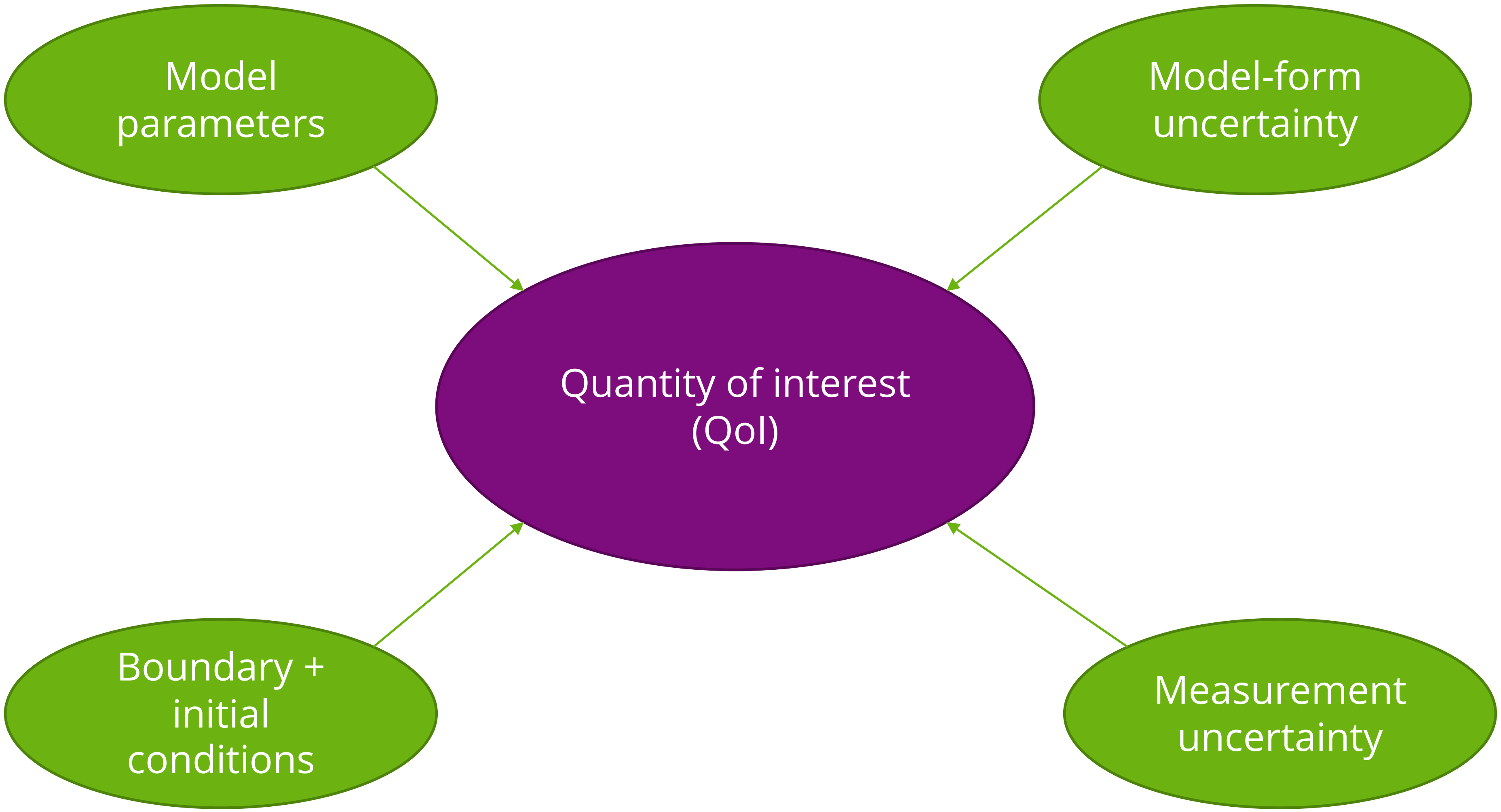
$$x(0) = 1, \qquad \frac{dx(t)}{dt}(0) = 0$$

Common sources of uncertainty





Result: uncertainty in model predictions



Real-world example

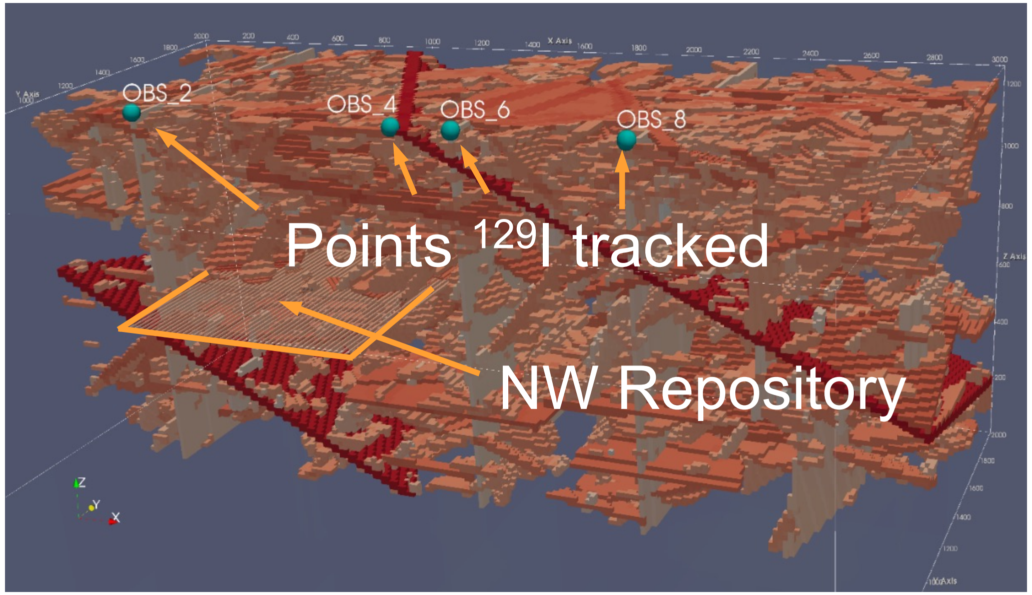
Quantity of Interest

- Concentration of radionuclide ^{129}I in nearby aquifer after 10^6 years

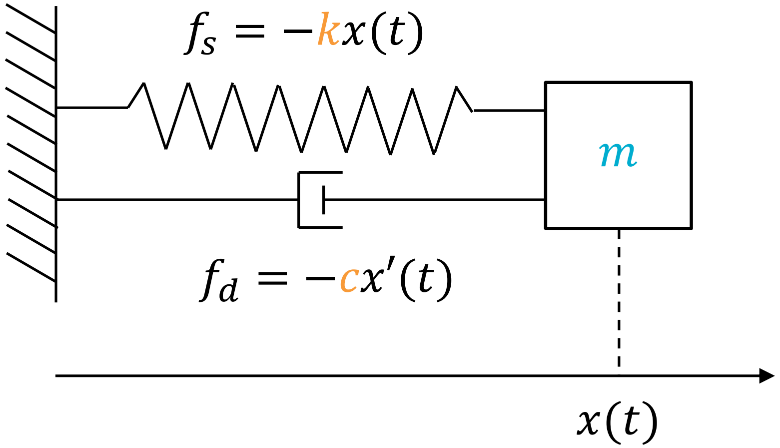
Sources of uncertainty

- Properties of canisters holding nuclear waste (e.g. degradation rate)
- Subsurface properties (e.g. porosity, permeability)
- Environmental conditions (e.g. incidence of earthquake, glaciation)

Nuclear waste repository modeling

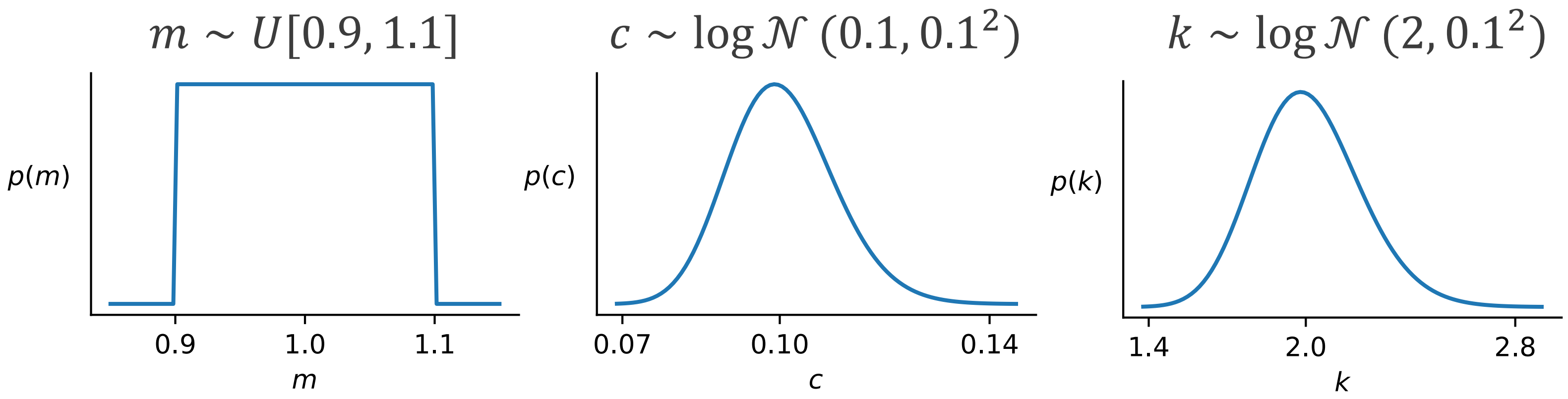


How do we ***characterize*** uncertainty?



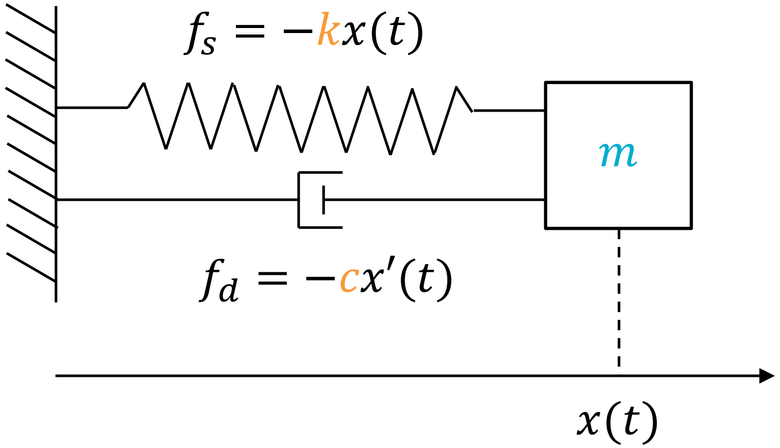
Represent sources of uncertainty as random variables (RVs)

Encode what is known through the parameterization of the RV

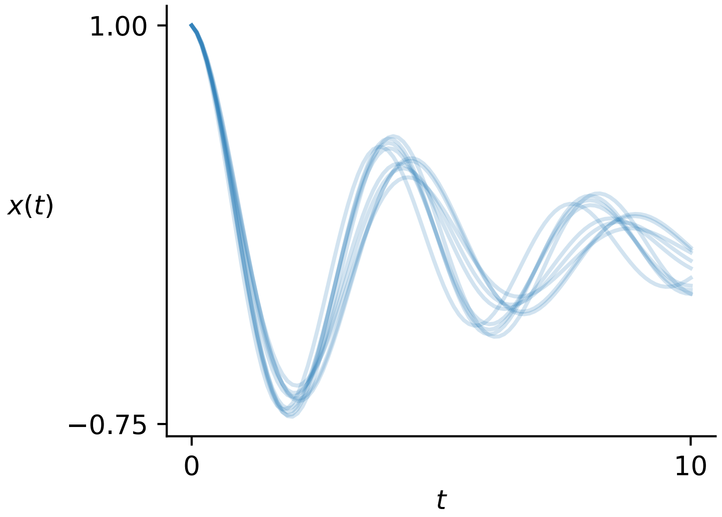


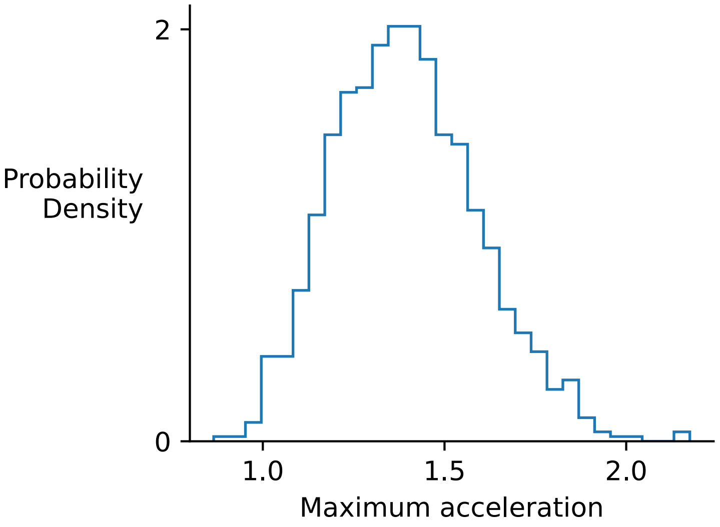
$$p_{\log\mathcal{N}(\mu,\sigma^2)}(x) \equiv \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(\log(x) - \mu)^2\right)$$

How do we ***quantify*** uncertainty?



Propagate sources of uncertainty to QoIs





Compute statistics of QoIs, e.g. mean, variance, tail probabilities

How do we ***reduce*** uncertainty?

Use data to gain more precise knowledge of sources of uncertainty

$$m\frac{d^2x(t)}{dt^2} + c\frac{dx(t)}{dt} + kx(t) = 0$$

$$d = x_{true}(t = 10) + \epsilon_m, \qquad \epsilon_m \sim \mathcal{N}(0, \sigma^2)$$

What is the **likelihood** the model produced the data for a given k, c?

$$\mathcal{M}(k, c) \equiv x(t = 10; k, c)$$

$$d = \mathcal{M}(k, c) + \epsilon_m, \qquad \epsilon_m \sim \mathcal{N}(0, \sigma^2)$$

$$d - \mathcal{M}(k, c) \sim \mathcal{N}(0, \sigma^2)$$

$$p(d|k, c) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{\left(d - M(k, c)\right)^2}{2\sigma^2}\right)$$

How do we ***reduce*** uncertainty?

Use data to gain more precise knowledge of sources of uncertainty

$$p(d|k,c) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{\left(d - M(k,c)\right)^2}{2\sigma^2}\right)$$

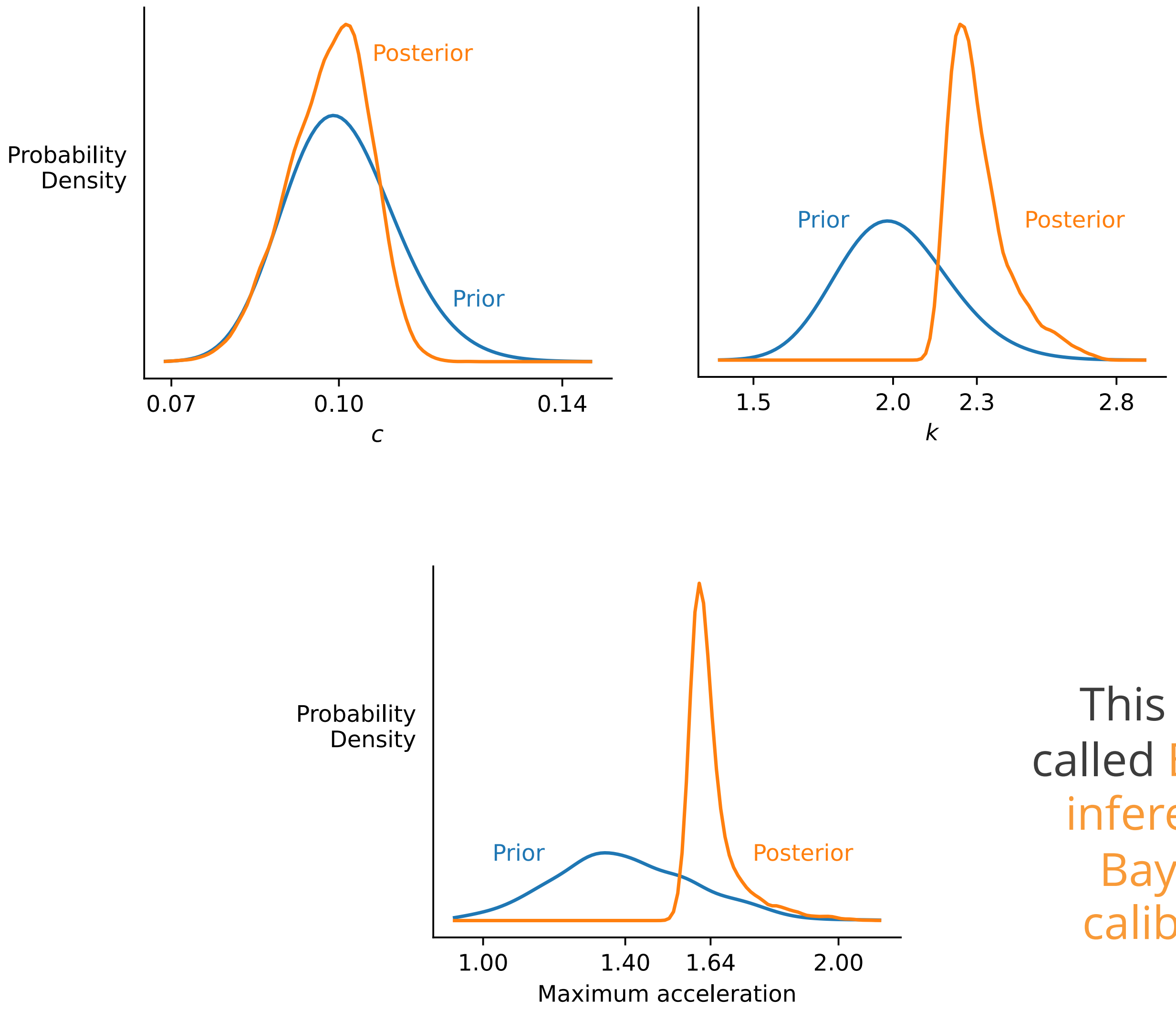
$$p(k,c) = p(k)p(c)$$

Bayes' Theorem

$$p(k,c|d) = \frac{p(d|k,c)p(k,c)}{\int p(d|k,c)p(k,c)\mathrm{d}k\mathrm{dc}}$$

How do we ***reduce*** uncertainty?

Use data to gain more precise knowledge of sources of uncertainty



This is also called Bayesian inference or Bayesian calibration

How do we

- characterize
- quantify
- reduce

uncertainty **in practice?**

Harsh reality

- Models for practical problems challenging
  - Nonlinear: propagating uncertainty + performing inference need many model evaluations
  - Computationally expensive; can afford few evaluations, causing poor statistical accuracy
  - High-dimensional problems

- Data expensive or impossible to attain

- Models imperfect representations of reality
  - Leads to unquantified error in predictions, biased Bayesian calibrations
  - But the correct model form is generally unknown (model-form uncertainty)

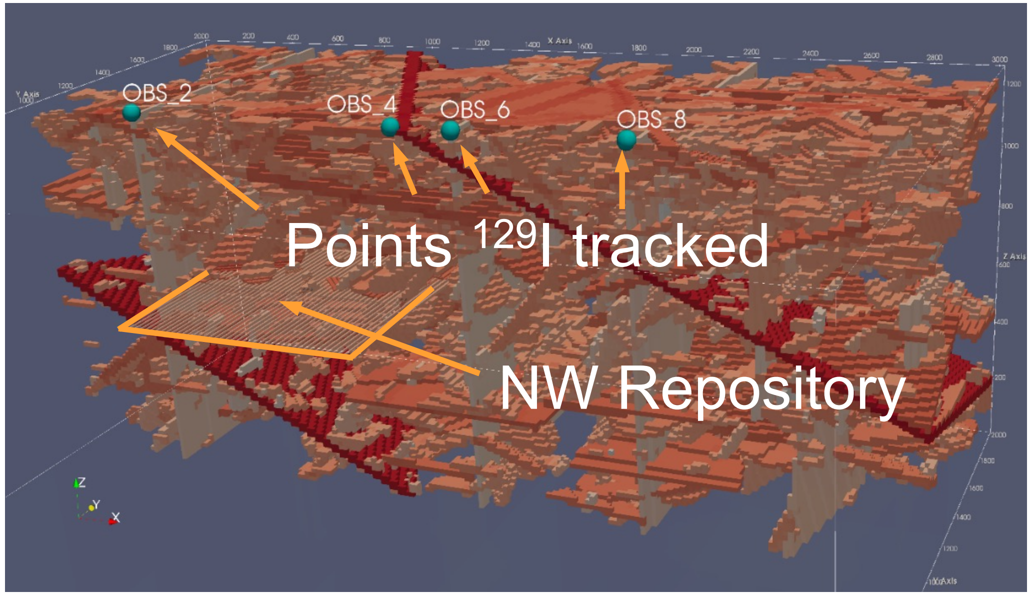
- Input/data uncertainties have to be modeled
  - If no quantitative information, have to encode prior belief through expert elicitation [4]

Real-world example

Data acquisition
- Subsurface properties: drill borehole(s)
- Canister properties: fabricate and test several canister specimens in the lab

Computational cost per model evaluation
- ~1.5 hours on 512 cores
- ~8 days on 4 cores
- ~22 years for 1000 samples on 4 cores

Nuclear waste repository modeling



Research areas in UQ

- Reduced-order/surrogate models
- Bayesian inverse problems
- Multimodel methods
- Optimal experimental design
- Sensitivity analysis
- Algorithms for high dimensionality
- Model-form uncertainty

Goal: reduce computational burden of UQ by using approximate representations of model

$$M(\theta) \approx f(\theta)$$

Common statistical approaches

Gaussian processes [5,6]

Stochastic expansions (polynomial chaos, stochastic collocation) [7]

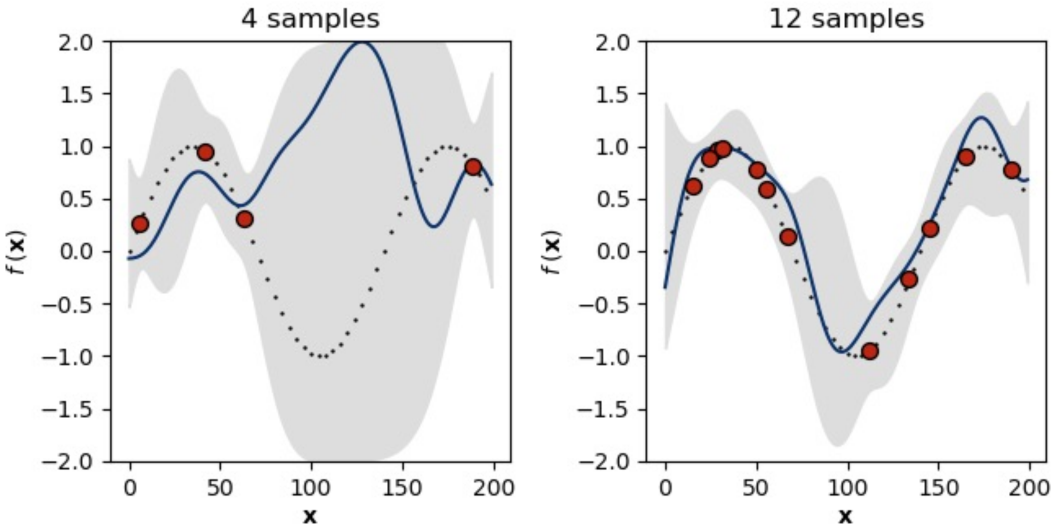
Common reduced-order model (ROM) approaches

Proper orthogonal decomposition [8,9]

Principal component analysis [10]

And, more recently

Machine learning!



https://gregorygundersen.com/blog/2019/06/27/gp-regression/

Ongoing opportunities

- Efficient surrogates/ROMs for high-dimensional models
- Adaptive/goal-oriented surrogate/ROM construction
- More theory for error incurred in UQ analyses by using approximation of $M(\theta)$
- Multimodel surrogates

Research areas in UQ

- Efficient forward propagation
- Bayesian inverse problems
- Reduced-order/surrogate models
- Optimal experimental design
- Sensitivity analysis
- Algorithms for high dimensionality
- Model-form uncertainty

Two types of high dimensionality

Infinite dimensionality

High cardinality

Ongoing opportunities

- Dimension reduction (PCA [10], active subspaces [11], autoencoders [12], ISOMAP [13])
  - Methods encouraging/exploiting sparsity

- Expanded theory for infinite-dimensional problems with less restrictive assumptions (e.g. linearity, Gaussianity) [14,15]

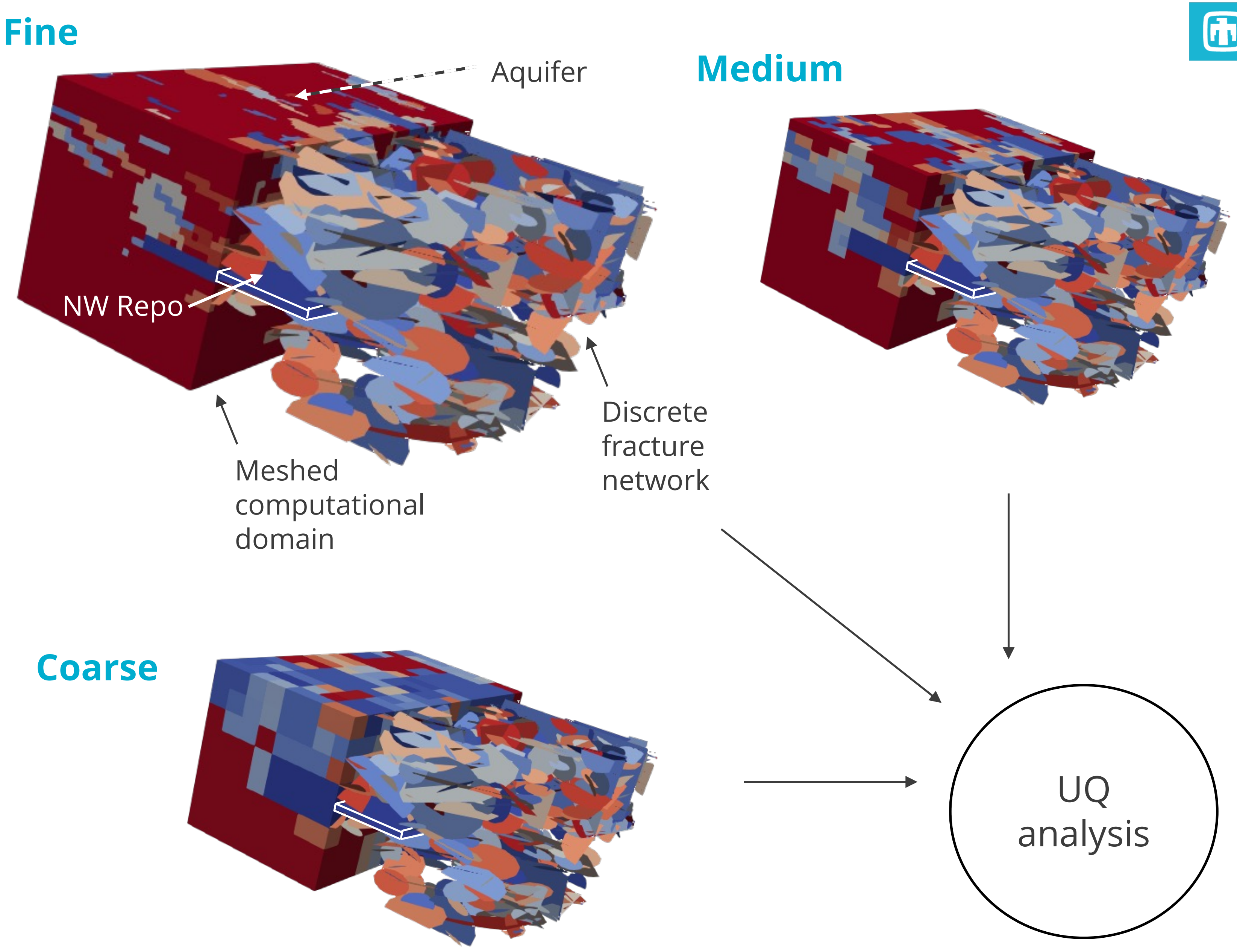
- Improve on existing inference methods
  - Derivative-based (MALA/HMC, Stochastic Newton, VI) [16-19]
  - Data-informed (DILI) [20]

- Methods to address high cardinality, especially for
  - Surrogates
  - Bayesian inference
  - Sensitivity analysis

Research areas in UQ

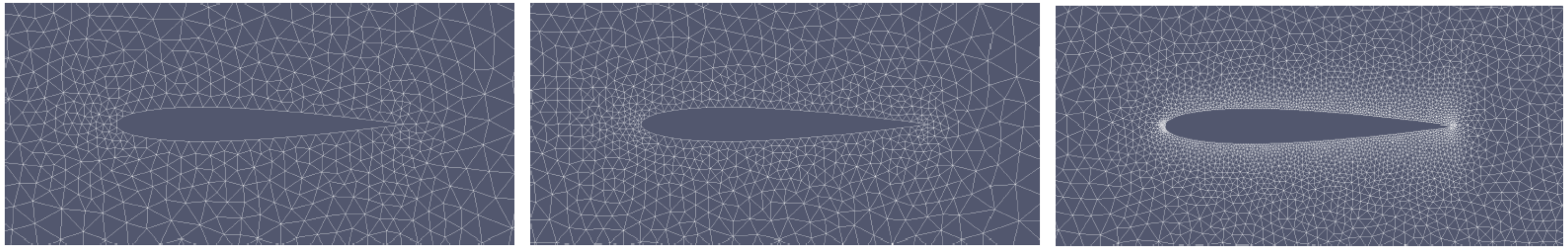
- Reduced-order/surrogate models
- Bayesian inverse problems
- Multimodel methods
- Optimal experimental design
- Sensitivity analysis
- Algorithms for high dimensionality
- Model-form uncertainty

Idea: exploit lower-fidelity, cheaper models to lower cost for same accuracy

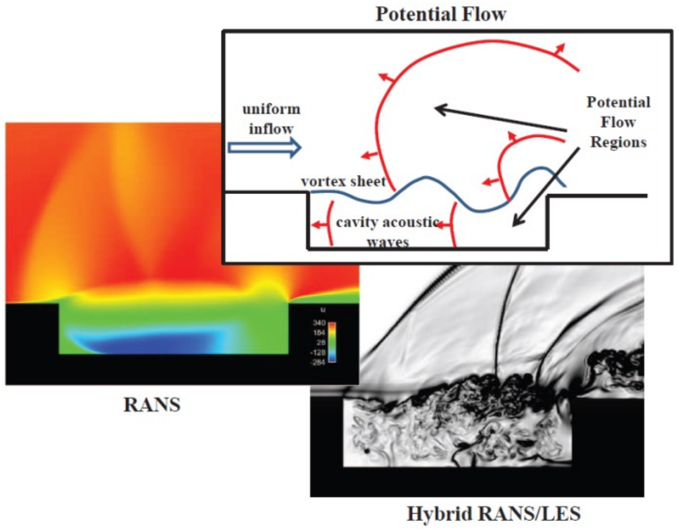


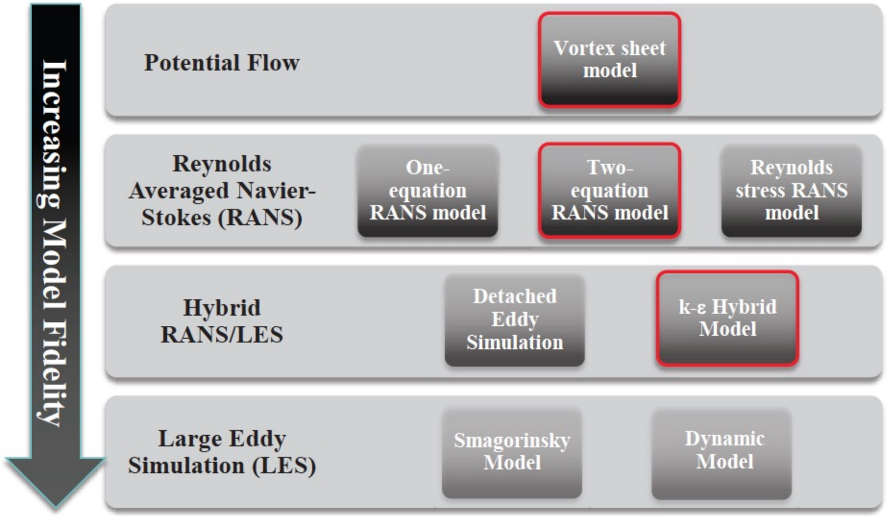
Idea: exploit lower-fidelity, cheaper models to lower cost for same accuracy

Discretization



Modeling assumptions





Sampling-based methods

$$\widehat{M}(\theta) = \frac{1}{N}\sum_{i=1}^{N} M\left(\theta^{(i)}\right), \qquad \theta^{(i)} \sim p(\theta) \text{ i.i.d.}$$

$$\mathbb{V}\left[\widehat{M}\right] = \frac{\mathbb{V}[M]}{N}$$

Sampling-based methods – control variates

$$M_1(\theta) \qquad c_1 = \frac{C_1}{C} \ll 1 \qquad \mathrm{corr}(\mathrm{M}, \mathrm{M}_1) = \rho$$

$$\widehat{M}_{CV}(\theta) = \widehat{M}(\theta) + \alpha\left(\widehat{M}_1(\theta) - \mathbb{E}[M_1]\right)$$

$$\mathbb{E}\left[\widehat{M}_{CV}(\theta)\right] = \mathbb{E}[M]$$ ← Unbiased

$$\mathbb{V}\left[\widehat{M}_{CV}(\theta)\right] = \frac{1}{N}\left(\mathbb{V}[M] + \alpha^2\mathbb{V}[M_1] + 2\alpha\mathrm{Cov}[M, M_1]\right)$$

$$\alpha^* = \min_{\alpha} \mathbb{V}\left[\widehat{M}_{CV}(\theta)\right] = -\frac{\mathrm{Cov}[M, M_1]}{\mathbb{V}[M_1]}$$

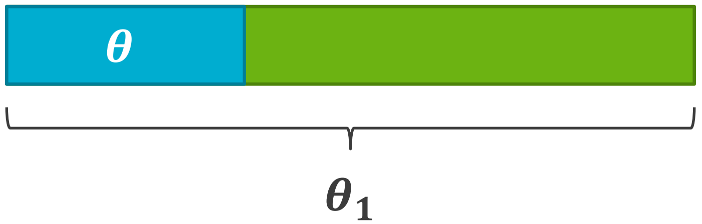
$$\mathbb{V}\left[\widehat{M}_{CV}(\theta)\right] = \frac{\mathbb{V}[M]}{N}(1 - \rho^2)$$ ← $\rho^2 \approx 1 \rightarrow$ orders of magnitude reduction in variance

Sampling-based methods – beyond control variates

$$\widehat{M}_{CV}(\theta) = \widehat{M}(\theta) + \alpha\left(\widehat{M}_1(\theta) - \mathbb{E}[M_1]\right)$$ Have to estimate this too

Multifidelity Monte Carlo [21]:

$$\widehat{M}_{MFMC} = \widehat{M}(\boldsymbol{\theta}) + \alpha\left(\widehat{M_1}(\boldsymbol{\theta}) - \widehat{M_1}(\boldsymbol{\theta}_1)\right)$$



$$\alpha^* = -\frac{\rho\sqrt{\mathbb{V}[M]}}{\sqrt{\mathbb{V}[M_1]}}$$

$$r_1^* = \sqrt{\frac{\text{Cost}(M)\rho^2}{\text{Cost}(M_1)(1-\rho^2)}} \qquad N_1 = \lceil r_1 N \rceil$$

$$\mathbb{V}\left[\widehat{M}_{MFMC}\right] = \frac{\mathbb{V}[M]}{N}\left(1 - \rho^2\left(\frac{r_1 - 1}{r_1}\right)\right)$$

Multimodel methods

- Theory extends to multiple (nonhierarchical) models
- Many algorithms combining different models and sample sets in different ways [21-24]
- Recent focus on multifidelity surrogates, e.g. Gaussian processes [25], multifidelity polynomial chaos [26], and several others [27-29]

$$M(\theta) \approx f(\theta)$$

vs.

$$M_1(\theta) \approx f_1(\theta)$$

$$M(\theta) - M_1(\theta) \approx f_\Delta(\theta)$$

$$M(\theta) \approx f_1(\theta) + f_\Delta(\theta)$$

Multimodel methods: ongoing opportunities

- Moving beyond functions of moments, e.g. tail probabilities, CDFs

- Startup cost of sampling all models to compute sample correlations → exploration vs exploitation tradeoff to find optimal model ensemble

- Stochastic models: stochasticity weakens correlation, but averaging it out can models too costly

- Addressing dissimilar parametrization (high and low fidelity models don't have same uncertain parameters)

Research areas in UQ

- Reduced-order/surrogate models
- Bayesian inverse problems
- Multimodel methods
- Optimal experimental design
- Sensitivity analysis
- Algorithms for high dimensionality
- Model-form uncertainty

Challenges and ongoing opportunities

$$p(d|\boldsymbol{\theta}) = \frac{p(d|\boldsymbol{\theta})p(\boldsymbol{\theta})}{\int p(d|\boldsymbol{\theta})p(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta}}$$

$$p(d|\boldsymbol{\theta}) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{\left(d - M(\boldsymbol{\theta})\right)^2}{2\sigma^2}\right)$$

If $M(\boldsymbol{\theta})$ nonlinear, can't compute analytically.

Markov Chain Monte Carlo [14,16-18,19] and Variational Inference [19] methods numerically approximate $p(\boldsymbol{\theta}|d)$--need many model evaluations

Much work in multimodel, derivative-based, & surrogate/reduced-order modeling methods to make this tractable. Methods in optimization can be leveraged

Opportunities for improvement: methods addressing multimodal and/or non-Gaussian posteriors; high dimensionality; model error

Research areas in UQ

- Reduced-order/surrogate models
- Bayesian inverse problems
- Multimodel methods
- Optimal experimental design
- Sensitivity analysis
- Algorithms for high dimensionality
- Model-form uncertainty

What model inputs (parameters) most affect model predictions?

Sensitivity analysis methods provide a quantitative measure of output sensitivity to each input [30]

Extremely powerful tool in mathematical modeling. Supports

- *Scientific discovery/model interpretation* – increase understanding of relationships between inputs + their interactions and outputs
- *Dimension reduction* – parameters identified to not affect model predictions can be screened out of further uncertainty analysis
- *Model improvement* – resources can be focused on reducing uncertainties where they will have the most impact

A range of methods

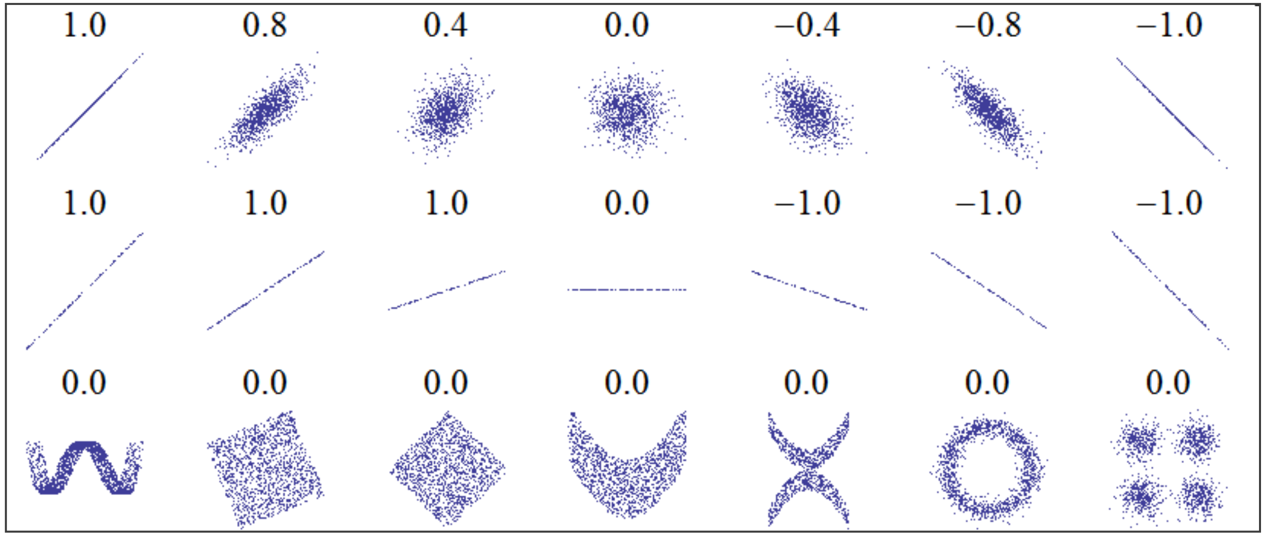
Correlation coefficients

$$\rho\big(\theta_i, M(\boldsymbol{\theta})\big) = \frac{\mathrm{Cov}[\theta_i, \mathrm{M}(\boldsymbol{\theta})]}{\sqrt{Var(\theta_i)Var\big(M(\boldsymbol{\theta})\big)}}$$

Estimated from input/output samples

Slope doesn't matter, just strength of linear relationship

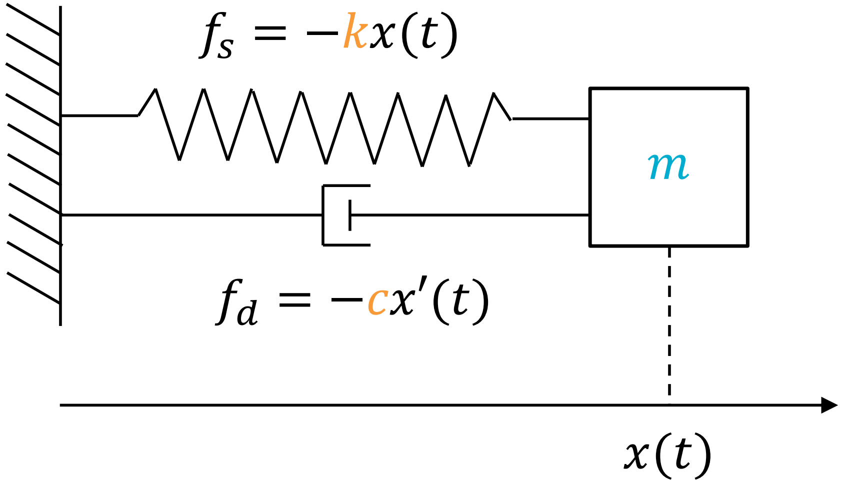
Nonlinear/nonmonotonic dependencies will not be detected.

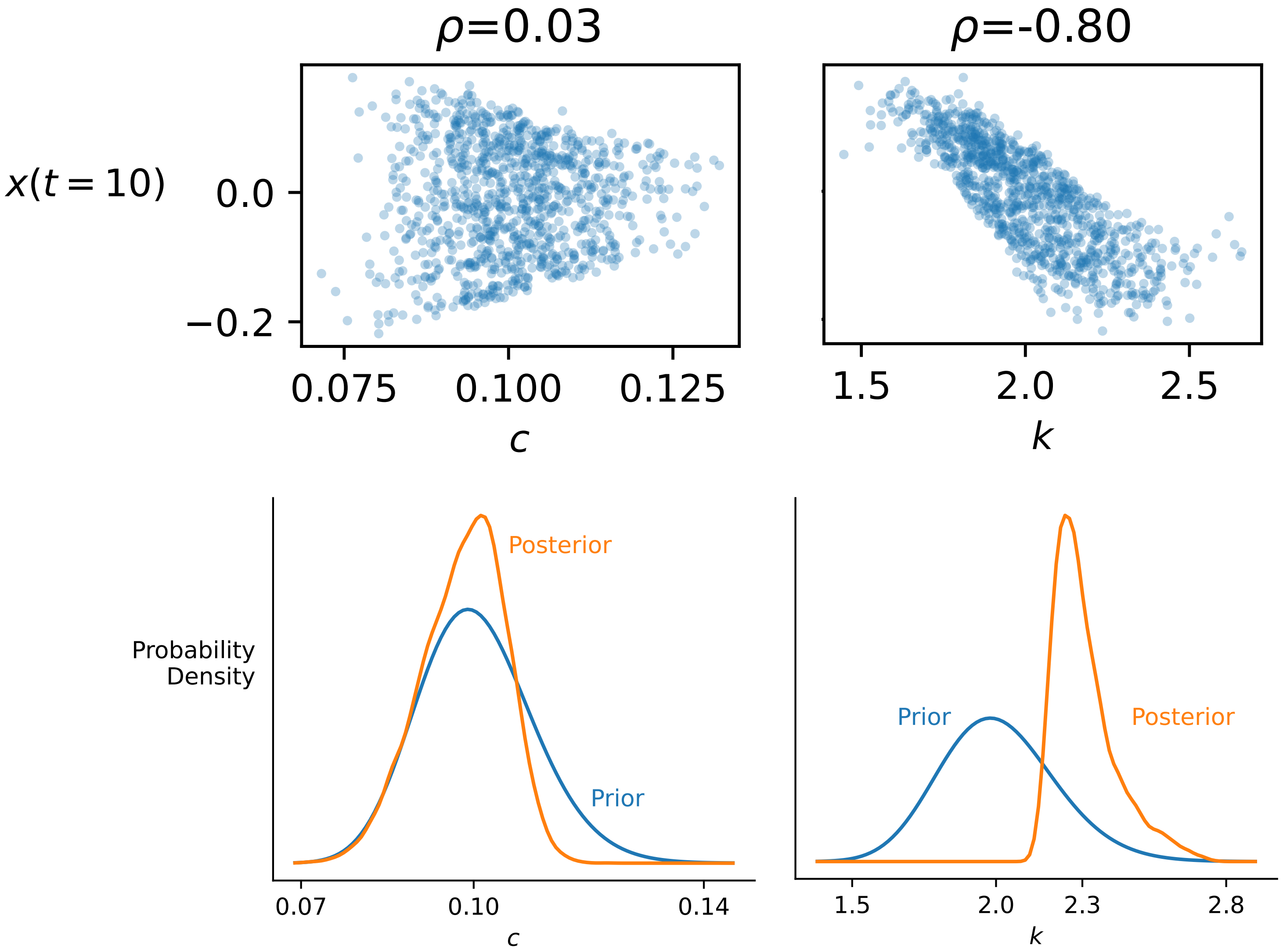


Source: http://en.wikipedia.org/wiki/Correlation

Higher-order dependencies (i.e. dependence on two parameters varying together) won't be detected.

Can detect which parameter(s) would be informed in calibration before even collecting data!





A range of methods

Global Variance-Based Sensitivity Analysis [30]

$$S_i = \frac{\mathbb{V}_{\theta_i}\left[\mathbb{E}_{\boldsymbol{\theta}_{\sim i}}[M|\theta_i]\right]}{\mathbb{V}[M]}$$

Effect of varying θ_i alone (averaging over other inputs)

$$T_i = 1 - \frac{\mathbb{V}_{\boldsymbol{\theta}_{\sim i}}\left[\mathbb{E}_{\theta_i}[M|\boldsymbol{\theta}_{\sim i}]\right]}{\mathbb{V}[M]}$$

Effect of varying θ_i alone and with all other inputs

Robust to nonlinearities and higher-order interactions between parameters

model evaluations: $N(d+2)$, N independent samples, d-dimensional input space

Assumes inputs statistically independent

A range of methods

- Distribution-based method [31]
  - Instead measure sensitivity of model output *distribution.*
  - Requires distribution to be estimated—extremely challenging with high input dimension

- Shapley values [32]
  - Game-theory based method
  - Relaxes assumption of independent inputs
  - Computationally costly ($2^d - 1$ evaluations)

Ongoing opportunities [33]

- Computational cost high for more advanced methods, $\mathcal{O}(d^{\alpha}), \alpha \geq 1$
- Computationally tractable methods for correlated inputs
- Unifying process to identify appropriate sensitivity method for a given task/goal

Research areas in UQ

- Efficient forward propagation
- Bayesian inverse problems
- Reduced-order/surrogate models
- Optimal experimental design
- Sensitivity analysis
- Algorithms for high dimensionality
- Model-form uncertainty

Bayesian OED overview

minimize uncertainty in parameter estimates

$d(w)$

$$\min_w \Psi(w) = f(p(\theta|d(w)))$$

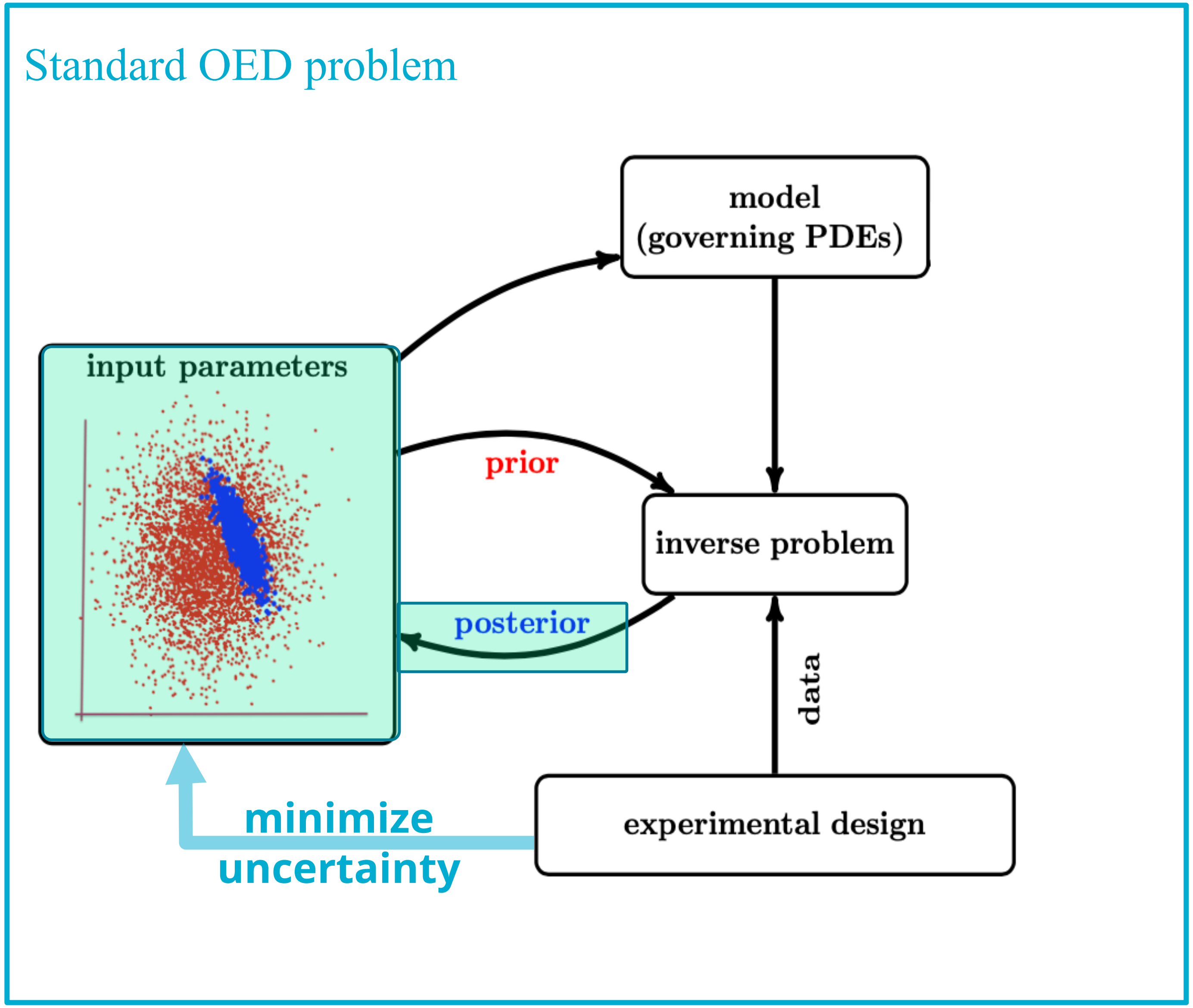


Figure courtesy of [34]

Slide courtesy of Rebekah White

Ongoing opportunities

- Outer-loop analysis on expensive Bayesian inverse problem; leverage all efficiency gains possible
  - Surrogates/ROMs
  - Multimodel methods
  - Dimension reduction
  - Derivative-based methods
- Methods to efficiently search experimental design space (especially if it's high dimensional, e.g. many sensors)
- Methods to address heterogeneous data (i.e. sensor and satellite image data)
- Goal-oriented approaches [35]

Goal-oriented OED overview

$d(w)$ $\min_{w} \Psi^{G}(w) = f(p(M(\theta)|d(w)))$

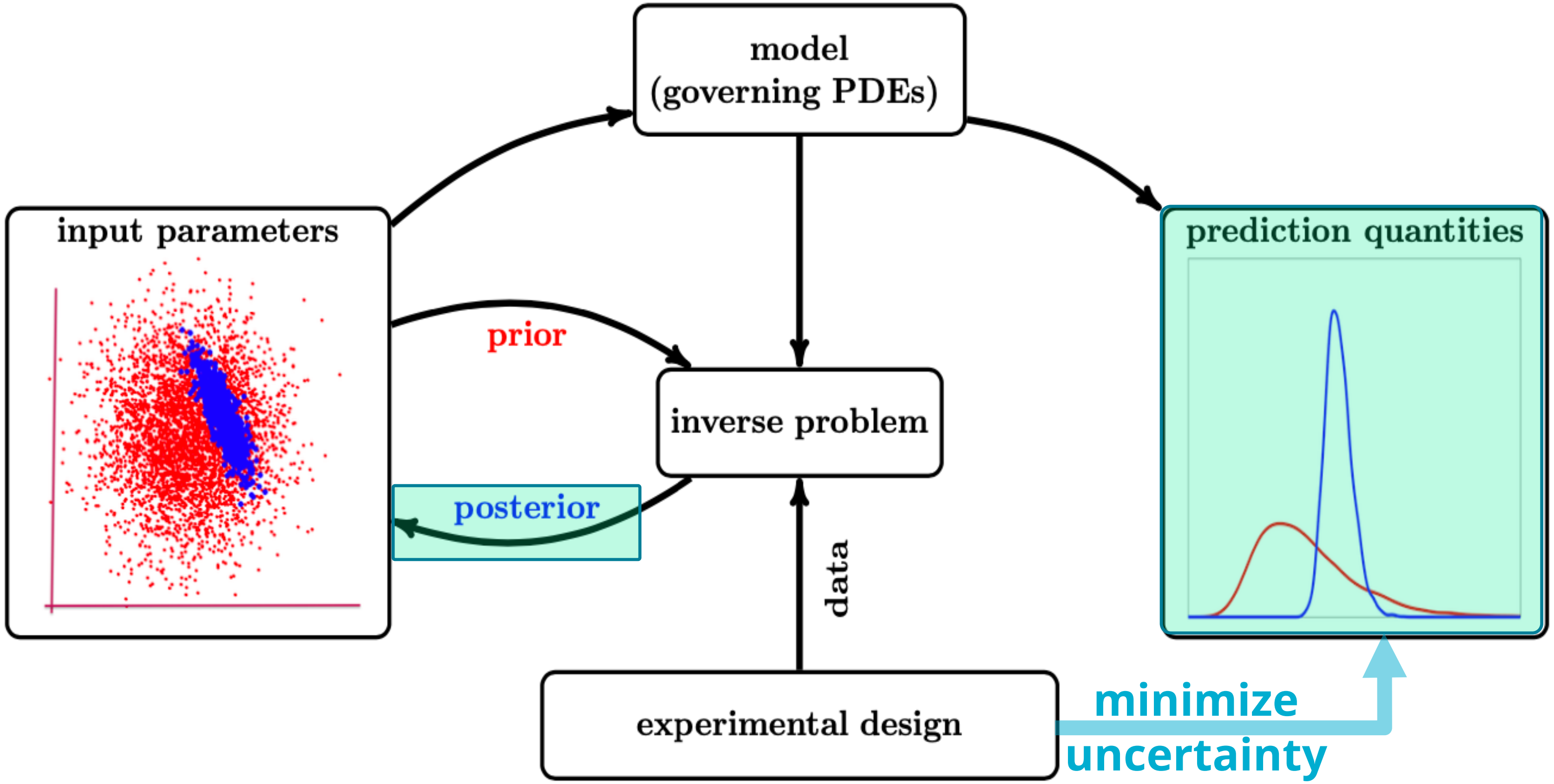


Figure courtesy of [34]

minimize uncertainty in predictions

Slide courtesy of Rebekah White

Research areas in UQ

Efficient forward propagation

Bayesian inverse problems

Reduced-order/surrogate models

Optimal experimental design

Sensitivity analysis

Algorithms for high dimensionality

Model-form uncertainty

Come to my talk tomorrow!

Thanks!

teresaportone.com

References

1. Smith, Ralph C. *Uncertainty Quantification: Theory, Implementation, and Applications*. Society for Industrial and Applied Mathematics, 2013.
2. Ghanem, Roger, David Higdon, and Houman Owhadi. *Handbook of Uncertainty Quantification*. Vol. 6. Springer, 2017. https://doi.org/10.1007/978-3-319-12385-1.
3. Dalbey, Keith, Michael Eldred, Gianluca Geraci, John Jakeman, Kathryn Maupin, Jason A Monschke, Daniel Seidl, Anh Tran, Friedrich Menhorn, and Xiaoshu Zeng. "Dakota A Multilevel Parallel Object-Oriented Framework for Design Optimization Parameter Estimation Uncertainty Quantification and Sensitivity Analysis: Version 6.16 Theory Manual." Sandia National Lab.(SNL-NM), Albuquerque, NM (United States), 2022. https://dakota.sandia.gov/sites/default/files/docs/6.16.0/Theory-6.16.0.pdf.
4. Meyer, Mary A, and Jane M Booker. *Eliciting and Analyzing Expert Judgment: A Practical Guide*. SIAM, 2001. https://doi.org/10.1137/1.9780898718485.
5. Gramacy, Robert B. Surrogates: Gaussian Process Modeling, Design, and Optimization for the Applied Sciences. Chapman and Hall/CRC, 2020. https://bobby.gramacy.com/surrogates/.
6. Rasmussen, C.E., and C.K.I. Williams. Gaussian Processes for Machine Learning. Adaptive Computation and Machine Learning Series. MIT Press, 2005. https://gaussianprocess.org/gpml/.
7. Xiu, Dongbin. *Numerical Methods for Stochastic Computations*. Princeton University Press, 2010. https://doi.org/10.1515/9781400835348.
8. Benner, Peter, Serkan Gugercin, and Karen Willcox. 2015. "A Survey of Projection-Based Model Reduction Methods for Parametric Dynamical Systems." SIAM Review 57 (4): 483–531. https://doi.org/10.1137/130932715.
9. Peherstorfer, Benjamin, and Karen Willcox. 2016. "Data-Driven Operator Inference for Nonintrusive Projection-Based Model Reduction." Computer Methods in Applied Mechanics and Engineering 306 (July): 196–215. https://doi.org/10.1016/j.cma.2016.03.025.
10. Jolliffe, Ian T, and Jorge Cadima. 2016. "Principal Component Analysis: A Review and Recent Developments." *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* 374 (2065): 20150202. https://doi.org/10.1098/rsta.2015.0202.
11. Constantine, Paul G. 2015. *Active Subspaces: Emerging Ideas for Dimension Reduction in Parameter Studies*. SIAM. https://doi.org/10.1137/1.9781611973860.
12. Wang, Yasi, Hongxun Yao, and Sicheng Zhao. 2016. "Auto-Encoder Based Dimensionality Reduction." *RoLoD: Robust Local Descriptors for Computer Vision 2014* 184 (April): 232–42. https://doi.org/10.1016/j.neucom.2015.08.104.
13. Tenenbaum, Joshua B., Vin de Silva, and John C. Langford. 2000. "A Global Geometric Framework for Nonlinear Dimensionality Reduction." *Science* 290 (5500): 2319–23. https://doi.org/10.1126/science.290.5500.2319.
14. Bui-Thanh, Tan, Omar Ghattas, James Martin, and Georg Stadler. 2013. "A Computational Framework for Infinite-Dimensional Bayesian Inverse Problems Part I: The Linearized Case, with Application to Global Seismic Inversion." SIAM Journal on Scientific Computing 35 (6): A2494–2523. https://doi.org/10.1137/12089586X.
15. Stuart, A. M. 2010. "Inverse Problems: A Bayesian Perspective." Acta Numerica 19: 451–559. https://doi.org/10.1017/S0962492910000061.
16. Girolami, Mark, and Ben Calderhead. 2011. "Riemann Manifold Langevin and Hamiltonian Monte Carlo Methods." *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 73 (2): 123–214. https://doi.org/10.1111/j.1467-9868.2010.00765.x.
17. Hoffman, Matthew D, and Andrew Gelman. 2014. "The No-U-Turn Sampler: Adaptively Setting Path Lengths in Hamiltonian Monte Carlo." J. Mach. Learn. Res. 15 (1): 1593–1623. https://www.jmlr.org/papers/volume15/hoffman14a/hoffman14a.pdf.
18. Petra, Noemi, James Martin, Georg Stadler, and Omar Ghattas. 2014. "A Computational Framework for Infinite-Dimensional Bayesian Inverse Problems, Part II: Stochastic Newton MCMC with Application to Ice Sheet Flow Inverse Problems." *SIAM Journal on Scientific Computing* 36 (4): A1525–55. https://doi.org/10.1137/130934805.
19. Blei, David M., Alp Kucukelbir, and Jon D. McAuliffe. 2017. "Variational Inference: A Review for Statisticians." Journal of the American Statistical Association 112 (518): 859–77. https://doi.org/10.1080/01621459.2017.1285773.
20. Cui, Tiangang, Kody J.H. Law, and Youssef M. Marzouk. 2016. "Dimension-Independent Likelihood-Informed MCMC." *Journal of Computational Physics* 304 (January): 109–37. https://doi.org/10.1016/j.jcp.2015.10.008.

References

21. Peherstorfer, Benjamin, Karen Willcox, and Max Gunzburger. “Optimal Model Management for Multifidelity Monte Carlo Estimation.” *SIAM Journal on Scientific Computing* 38, no. 5 (January 1, 2016): A3163–94. https://doi.org/10.1137/15M1046472.
22. Giles, Michael B. “Multilevel Monte Carlo Methods.” *Acta Numerica* 24 (2015): 259–328. https://doi.org/10.1017/S096249291500001X.
23. Gorodetsky, Alex A., Gianluca Geraci, Michael S. Eldred, and John D. Jakeman. “A Generalized Approximate Control Variate Framework for Multifidelity Uncertainty Quantification.” *Journal of Computational Physics* 408 (May 1, 2020): 109257. https://doi.org/10.1016/j.jcp.2020.109257.
24. Schaden, Daniel, and Elisabeth Ullmann. “On Multilevel Best Linear Unbiased Estimators.” *SIAM/ASA Journal on Uncertainty Quantification* 8, no. 2 (January 1, 2020): 601–35. https://doi.org/10.1137/19M1263534.
25. Kennedy, M. C., and A. O’Hagan. 2000. “Predicting the Output from a Complex Computer Code When Fast Approximations Are Available.” Biometrika 87 (1): 1–13. http://www.jstor.org/stable/2673557.
26. Ng, Leo Wai-Tsun, and Michael Eldred. 2012. “Multifidelity Uncertainty Quantification Using Non-Intrusive Polynomial Chaos and Stochastic Collocation.” In 53rd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference. Structures, Structural Dynamics, and Materials and Co-Located Conferences. American Institute of Aeronautics and Astronautics. https://doi.org/10.2514/6.2012-1852.
27. Teckentrup, A. L., P. Jantsch, C. G. Webster, and M. Gunzburger. 2015. “A Multilevel Stochastic Collocation Method for Partial Differential Equations with Random Input Data.” SIAM/ASA Journal on Uncertainty Quantification 3 (1): 1046–74. https://doi.org/10.1137/140969002.
28. Jakeman, John D., Michael S. Eldred, Gianluca Geraci, and Alex Gorodetsky. 2020. “Adaptive Multi-Index Collocation for Uncertainty Quantification and Sensitivity Analysis.” International Journal for Numerical Methods in Engineering 121 (6): 1314–43. https://doi.org/10.1002/nme.6268.
29. Gorodetsky, A. A., J. D. Jakeman, and G. Geraci. 2021. “MFNets: Data Efficient All-at-Once Learning of Multifidelity Surrogates as Directed Networks of Information Sources.” Computational Mechanics 68 (4): 741–58. https://doi.org/10.1007/s00466-021-02042-0.
30. Saltelli, Andrea, Chan, K., and Scott, E.M. *Sensitivity Analysis*. Wiley, 2009.
31. Borgonovo, E. “A New Uncertainty Importance Measure.” *Reliability Engineering & System Safety* 92, no. 6 (June 1, 2007): 771–84. https://doi.org/10.1016/j.ress.2006.04.015.
32. Owen, Art B., and Clémentine Prieur. “On Shapley Value for Measuring Importance of Dependent Inputs.” *SIAM/ASA Journal on Uncertainty Quantification* 5, no. 1 (January 1, 2017): 986–1002. https://doi.org/10.1137/16M1097717.
33. Razavi, Saman, Anthony Jakeman, Andrea Saltelli, Clémentine Prieur, Bertrand Iooss, Emanuele Borgonovo, Elmar Plischke, et al. “The Future of Sensitivity Analysis: An Essential Discipline for Systems Modeling and Policy Support.” *Environmental Modelling & Software* 137 (March 1, 2021): 104954. https://doi.org/10.1016/j.envsoft.2020.104954.
34. Alexanderian, Alen. 2021. “Optimal Experimental Design for Infinite-Dimensional Bayesian Inverse Problems Governed by PDEs: A Review.” *Inverse Problems* 37 (4): 043001. https://doi.org/10.1088/1361-6420/abe10c.
35. Attia, Ahmed, Alen Alexanderian, and Arvind K Saibaba. 2018. “Goal-Oriented Optimal Design of Experiments for Large-Scale Bayesian Linear Inverse Problems.” *Inverse Problems* 34 (9): 095009. https://doi.org/10.1088/1361-6420/aad210.